An application for hyperbolic numbers

Z.

November 19, 2026

While generating solutions to a Diophantine equation, I stumbled upon a fascinating use for the hyperbolic numbers, one that displays a deep connection to algebraic number theory. To define the hyperbolic numbers, let $j^2 = 1 \quad (j \neq \pm 1)$. The hyperbolic numbers are then all numbers of the form $a+bj \quad (a, b \in \mathbb{R})$. Note that this is the same as $\mathbb{R}[j]$. The equation I was working on when I discovered this connection was the following:

$$a^2 + b^2 + c^2 = d^2 \quad (a, b, c, d \in \mathbb{Z})$$

To generate solutions, I found $(1, 4, 8, 9)$ by hand, then rearranged the equation into $a^2 + b^2 = d^2 - c^2$, which I factored over the complex numbers to get $(a + bi)(a - bi) = (d + c)(d - c)$. This is where I hit a roadblock. The usual method for generating solutions to an equation like this is quite simple. Since a and b are integers, $(a + bi) \in \mathbb{Z}[i]$. Now use the fact that $\mathbb{Z}[i]$ is closed under $(a+bi)^n \quad (1 \leq n, n \in \mathbb{Z})$, and that if $(a+bi)^n = u+vi$ then $(a-bi)^n = u-vi$. The same behavior applies to $\mathbb{Z}[\sqrt{k}]$ in general, so $(d + c)^n$ follows the same rules. This allowed me to find that $(a+bi)^n = (a_n+b_ni)$. The issue came when I tried to apply it to the other half: $(d + c)^n = (d_n + c_n)$ does indeed hold; however, there is no separation between the two integer parts, so they combine into the single integer $d_n + c_n$ when we need the pair (d_n, c_n). On its own, this one number is not enough, because there are many ways to reverse an addition $(10 = 5 + 5, 6 + 4, 7 + 3 \ldots)$. The information is not truly lost: the other factor gives $(d - c)^n = (d_n - c_n)$, and from the sum and the difference we can recover both parts,

$$d_n = \frac{(d + c)^n + (d - c)^n}{2} \qquad c_n = \frac{(d + c)^n - (d - c)^n}{2}$$

But this means carrying two separate integers around and untangling them at the end, where the left side needed only the single number $a+bi$. It is simple to see why this happens: when you factor a difference of squares, you are essentially factoring it over $\mathbb{Z}[\sqrt{1}]$, and because $\sqrt{1} \in \mathbb{Z}$, it follows that $\mathbb{Z}[\sqrt{1}] = \mathbb{Z}$. This is where the awkwardness comes from. If only there existed a number which was a square root of 1, but was explicitly *not* an integer.

$\mathbb{Z}[j]$ is the direct solution to this problem. In $\mathbb{Z}[j]$, $(d^2-c^2) = (d+cj)(d-cj)$. Unlike with the ordinary integers, however, $(d + cj)^n = (d_n + c_nj)$ does not collapse into a single number, meaning you can easily separate d_n and c_n by taking the real and hyperbolic parts. Now we can try to find nontrivial solutions to the equation. First, let's look at where the first attempt breaks:

$$(1 + 4i)^2 = (-15 + 8i)$$

$$a_2 = -15 \qquad b_2 = 8$$

$$(9+8)^2 = 289$$

Uh oh, we have a single number when we actually need a pair of numbers that we can split apart, as with the Gaussian integers. This is where the hyperbolic method finally shines:

$$a_2 = -15 \qquad b_2 = 8$$

$$(9+8j)^2 = 81 + 64 + 72j + 72j = (145 + 144j)$$

Wonderful! Now, recalling our original identity, $145 + 144j = (d_2 + c_2 j)$, so

$$c_2 = 144 \qquad d_2 = 145$$

Our new "Pythagorean quadruple" is $(-15, 8, 144, 145)$. Now we can check that the method holds (I will replace -15 with 15, since $15^2 = (-15)^2$):

$$15^2 + 8^2 + 144^2 = 145^2$$

This identity holds!